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Oscillatory flow of a fourth-order fluid

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SUMMARY

This study deals with the incompressible flow of a fourth-order fluid over a porous plate oscillating in its own plane. Numerical solution of the nonlinear problem governing the flow is given. The influence of various parameters of interest on the velocity distribution is shown and discussed with the help of several graphs. Copyright © 2008 John Wiley & Sons, Ltd.

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KEY WORDS: fourth-order fluid; oscillating plate; suction/injection; nonlinear; numerical solution; Newton's method

1. INTRODUCTION

Due to several industrial applications, the non-Newtonian fluids are considered more appropriate than the Newtonian fluids, especially during the last four decades. In fact, some rheological complex fluids, for example, shampoo, blood, paints, ketchup, paste, polymer solutions and certain oils cannot be adequately described by the Navier–Stokes theory. Due to great diversity in non-Newtonian fluids, it is not possible to recommend a single constitutive equation, which can describe all the properties of non-Newtonian fluids. In view of this various empirical or semi-empirical constitutive equations have been proposed. In spite of various constitutive equations, many questions are still not answered. Amongst the various constitutive equations, the fluids of the second grade are the simplest subclass of non-Newtonian fluids. Because of the Clausius–Duhem inequality and the assumption that the Helmholtz free energy is minimum in equilibrium, the three material constants in the constitutive equation of second-grade fluid are restricted. A detailed excellent discussion on this issue is made by Dunn and Rajagopal [1], Rajagopal [2] and Fosdick and Rajagopal [3]. Such a discussion for the third-grade fluid model is given by Fosdick and Rajagopal [4].

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Extensive research has been undertaken for unidirectional flows of a second-grade fluid (simplest subclass of a differential-type fluid). This is perhaps due to the fact that in second-grade fluid, the governing equation for unidirectional flow is linear whereas it is nonlinear in third- and fourth-order fluids. It is now known that the steady unidirectional flows of a second-grade fluid over rigid boundaries do not include the rheological characteristics in the solution. Because of this fact the third- and fourth-order models have gained much importance. Such models include the rheological properties even for the steady unidirectional flows over rigid boundaries. Important contributions regarding the unidirectional flows of viscoelastic fluids are given in References [5–14]. It is known that in general the governing equations for the non-Newtonian fluids are of higher order than the Navier–Stokes equations and thus the adherence conditions become insufficient. Therefore, the critical review regarding the boundary conditions, the existence and uniqueness of the solution given by Rajagopal [15, 16], Rajagopal *et al.* [17] and Rajagopal and Kaloni [18] is very important. Rajagopal and Sciubba [19] also discussed the important analysis of pulsating Poiseuille flow of a non-Newtonian fluid.

This study is undertaken to investigate the flow of fourth-grade fluid over a porous plate. The porous plate is assumed to induce oscillations in its own plane. The numerical solution of the arising nonlinear problem is obtained using Newton's method. The effects of main emerging parameters on the velocity distribution are studied and also compared with those of Newtonian fluids. It is important to mention here that fluid models such as the one considered here involve too many material constants that characterize them and in fact it is not possible to measure these constants experimentally; moreover only values for certain combinations of the constants can be computed. That is why the thermodynamic studies of such models that help provide information regarding these constants are useful. Furthermore, the fluid model under consideration is incapable of stress relaxation and hence cannot describe the response of many viscoelastic fluids. However, the model can describe shear thinning, shear thickening and normal stress differences.

2. ANALYSIS

2.1. Basic equations

The equations expressing the incompressibility condition and the balance of linear momentum (in the absence of body forces) are given by

$$\operatorname{div} \mathbf{V} = 0 \tag{1}$$

$$\rho \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \operatorname{div} \mathbf{T} \tag{2}$$

where ρ is the fluid density, $\mathbf{V}$ is the velocity and $\mathrm{d}/\mathrm{d}t$ is the material derivative. For fourth-order fluid, the constitutive equation for the Cauchy stress $\mathbf{T}$ is the following [20]:

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \mathbf{S}_1 + \mathbf{S}_2 \tag{3}$$

$$\mathbf{S}_1 = \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_2) + \beta_3(\operatorname{tr}\mathbf{A}_1^2)\mathbf{A}_1 \tag{4}$$

$$\begin{aligned}\mathbf{S}_2 = {} & \gamma_1\mathbf{A}_4 + \gamma_2(\mathbf{A}_3\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}_3) + \gamma_3\mathbf{A}_2^2 + \gamma_4(\mathbf{A}_2\mathbf{A}_1^2 + \mathbf{A}_1^2\mathbf{A}_2) \\ & + \gamma_5(\operatorname{tr}\mathbf{A}_2)\mathbf{A}_2 + \gamma_6(\operatorname{tr}\mathbf{A}_2)\mathbf{A}_1^2 + (\gamma_7\operatorname{tr}\mathbf{A}_3 + \gamma_8\operatorname{tr}(\mathbf{A}_2\mathbf{A}_1))\mathbf{A}_1\end{aligned} \tag{5}$$

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where p is the hydrostatic pressure, $\mathbf{I}$ is the identity tensor, μ is the dynamic viscosity and α_i ($i=1,2$), β_i ($i=1$–3) and γ_i ($i=1$–8) are the material constants corresponding to second-, third- and fourth-order fluids, respectively. The kinematical tensors $\mathbf{A}_1$–$\mathbf{A}_4$ are the first four Rivlin–Ericksen tensors defined as

$$\mathbf{A}_1=\nabla\mathbf{V}+(\nabla\mathbf{V})^{\mathrm{T}} \tag{6}$$

$$\mathbf{A}_n=\frac{\mathrm{d}\mathbf{A}_{n-1}}{\mathrm{d}t}+\mathbf{A}_{n-1}(\nabla\mathbf{V})+(\nabla\mathbf{V})^{\mathrm{T}}\mathbf{A}_{n-1} \quad (n>1) \tag{7}$$

2.2. Problem formulation

Let us consider the flow of an incompressible fourth-order fluid with constant properties. The fluid is over an oscillating plate at $y=0$. The x-axis is chosen parallel to the plate. Moreover, the plate is porous and oscillates in its own plane. The flow is independent upon x (i.e. $u=u(y,t)$, u is the velocity in the x direction). Under these assumptions, the Equations (2)–(7) along with the continuity equation (1) give

$$\begin{aligned}
\rho\left[\frac{\partial u}{\partial t}+V_0\frac{\partial u}{\partial y}\right]&=\mu\frac{\partial^2 u}{\partial y^2}+\alpha_1\left[\frac{\partial^3 u}{\partial y^2\partial t}+V_0\frac{\partial^3 u}{\partial y^3}\right]\\
&+\beta_1\left[\frac{\partial^4 u}{\partial y^2\partial t^2}+2V_0\frac{\partial^4 u}{\partial y^3\partial t}+V_0^2\frac{\partial^4 u}{\partial y^4}\right]+6(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial^2 u}{\partial y^2}\right)\\
&+\gamma_1\left[\frac{\partial^5 u}{\partial y^2\partial t^3}+3V_0\frac{\partial^5 u}{\partial y^3\partial t^2}+3V_0^2\frac{\partial^5 u}{\partial y^4\partial t}+V_0^3\frac{\partial^5 u}{\partial y^5}\right]\\
&+2\begin{pmatrix}3\gamma_2+\gamma_3+\gamma_4\\ +\gamma_5+3\gamma_7+\gamma_8\end{pmatrix}\left[\begin{array}{l}2\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y^2}\right)\left(\frac{\partial^2 u}{\partial y\partial t}\right)\\ +\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial y^2\partial t}\\ +2V_0\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y^2}\right)\\ +V_0\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial^3 u}{\partial y^3}\right)\end{array}\right]
\end{aligned} \tag{8}$$

where $V_0<0$ corresponds to the suction case and $V_0>0$ for blowing and modified pressure has been neglected.

The relevant boundary conditions for the flow are

$$\begin{aligned}
u(0,t)&=U_0\mathrm{e}^{-\mathrm{i}\omega t}, \quad \omega>0, \ t>0\\
u(y,t)&\longrightarrow 0 \quad \text{as } y\longrightarrow\infty, \quad U(y,0)=0, \ y>0
\end{aligned} \tag{9}$$

where U_0 is the reference velocity and ω is the oscillating frequency. It is noticed that Equation (8) is fifth order and the available boundary conditions are insufficient to solve the problem. Such difficulty is removed by augmenting the boundary conditions at infinity by assuming certain asymptotic structure for the flow i.e.

$$\frac{\partial^n u}{\partial y^n} \longrightarrow 0 \quad \text{as } y \longrightarrow \infty \quad (n=1,2,3) \tag{10}$$

It should be pointed out that such a procedure of augmenting the boundary conditions is not possible if the flow takes place between two parallel plates a finite distance apart. A related interesting work dealing with the issue of boundary conditions in non-Newtonian and Navier–Stokes fluids is given in the Reference [21]. Expression for the shear stress is

$$\begin{aligned}\tau_{xy} = {} & \mu\left(\frac{\partial u}{\partial y}\right) + \alpha_1\left(\frac{\partial^2 u}{\partial y \partial t} + V_0 \frac{\partial^2 u}{\partial y^2}\right) + \beta_1\left(\frac{\partial^3 u}{\partial y \partial t^2} + 2V_0 \frac{\partial^3 u}{\partial y^2 \partial t} + V_0^2 \frac{\partial^3 u}{\partial y^3}\right) \\ & + 2(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^3 + \gamma_1\left(\frac{\partial^4 u}{\partial y \partial t^3} + 3V_0 \frac{\partial^4 u}{\partial y^2 \partial t^2} + 3V_0^2 \frac{\partial^4 u}{\partial y^3 \partial t} + V_0^3 \frac{\partial^4 u}{\partial y^4}\right) \\ & + 2(3\gamma_2+\gamma_3+\gamma_4+\gamma_5+3\gamma_7+\gamma_8)\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial^2 u}{\partial y \partial t} + V_0 \frac{\partial^2 u}{\partial y^2}\right)\end{aligned} \tag{11}$$

Introducing the following non-dimensional variables:

$$\eta = \sqrt{\frac{\omega}{2v}}\, y, \quad \tau = \omega t, \quad f = \frac{u}{U_0} \tag{12}$$

where v is the kinematic viscosity, we get

$$\begin{aligned}\left[\frac{\partial f}{\partial \tau} + \sqrt{2} d\left(\frac{\partial f}{\partial \eta}\right)\right] = {} & \frac{1}{2}\frac{\partial^2 f}{\partial \eta^2} + \frac{a}{2}\left(\frac{\partial^3 f}{\partial \eta^2 \partial \tau} + \sqrt{2} d \frac{\partial^3 f}{\partial \eta^3}\right) \\ & + b_1\left(\frac{\partial^4 f}{\partial \eta^2 \partial \tau^2} + d \frac{\partial^4 f}{\partial \eta^3 \partial \tau} + 2d^2 \frac{\partial^4 f}{\partial \eta^4}\right) \\ & + \frac{3}{2}(b_2+b_3)\left(\frac{\partial f}{\partial \eta}\right)^2\left(\frac{\partial^2 f}{\partial \eta^2}\right) \\ & + \frac{c_1}{2}\left(\begin{array}{l}\dfrac{\partial^5 f}{\partial \eta^2 \partial \tau^3} + 3\sqrt{2} d \dfrac{\partial^5 f}{\partial \eta^3 \partial \tau^2} \\ \qquad + 6d^2 \dfrac{\partial^5 f}{\partial \eta^5 \partial \tau} + \sqrt{2} d^3 \dfrac{\partial^5 f}{\partial \eta^5}\end{array}\right)\end{aligned}$$

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$$+\frac{1}{2}\begin{pmatrix}3c_2+c_3+c_4\\ \quad +c_5+3c_7+c_8\end{pmatrix}\begin{bmatrix}2\left(\frac{\partial f}{\partial \eta}\right)\left(\frac{\partial^2 f}{\partial \eta^2}\right)\left(\frac{\partial^2 f}{\partial \eta \partial \tau}\right)\\ +\left(\frac{\partial f}{\partial \eta}\right)^2\left(\frac{\partial^3 f}{\partial \eta^2 \partial \tau}\right)\\ +d\left(\frac{\partial f}{\partial \eta}\right)\left(\frac{\partial^2 f}{\partial \eta^2}\right)\\ +\sqrt{2}d\left(\frac{\partial f}{\partial \eta}\right)^2\left(\frac{\partial^3 f}{\partial \eta^3}\right)\end{bmatrix} \tag{13}$$

$$f(0,\tau)=\mathrm{e}^{-\mathrm{i}\tau}, \quad f(\eta,\tau)\longrightarrow 0 \quad \text{as } \eta\longrightarrow\infty, \quad f(\eta,0)=0 \tag{14}$$

where

$$\begin{aligned} &a=\frac{\alpha_1\omega}{\rho v}, \quad b_1=\frac{\beta_1\omega^2}{\rho v}, \quad b_2=\frac{\beta_2\omega U_0^2}{\rho v^2}, \quad b_3=\frac{\beta_3\omega U_0^2}{\rho v^2}\\ &d=\frac{V_0}{2\sqrt{v\omega}}, \quad c_1=\frac{\gamma_1\omega^3}{\rho v}\\ &c_2=c_3=c_4=c_5=c_7=c_8=\frac{\gamma_i\omega^2 U_0^2}{\rho v^2}, \quad i=2,3,4,5,7,8 \end{aligned} \tag{15}$$

3. NUMERICAL RESULTS AND DISCUSSION

We note that Equation (13) is a fifth-order partial differential equation. It is perhaps not possible to obtain the exact analytic solution. Due to this, we seek the numerical solution. For obtaining the system of algebraic equations we use the following approximations to the derivatives:

$$\frac{\partial f}{\partial \tau}=\frac{1}{k}(f_{i,j}-f_{i,j-1}) \tag{16}$$

$$\frac{\partial^2 f}{\partial \tau^2}=\frac{1}{k^2}(f_{i,j+1}-2f_{i,j}+f_{i,j-1}) \tag{17}$$

$$\frac{\partial f}{\partial \eta}=\frac{1}{2h}(f_{i+1,j}-f_{i-1,j}) \tag{18}$$

$$\frac{\partial^2 f}{\partial \eta^2}=\frac{1}{h^2}(f_{i+1,j}-2f_{i,j}+f_{i-1,j}) \tag{19}$$

$$\frac{\partial^3 f}{\partial \eta^3}=\frac{1}{2h^3}(f_{i+2,j}-2f_{i+1,j}+2f_{i-1,j}-f_{i-2,j}) \tag{20}$$

$$\frac{\partial^4 f}{\partial\eta^4}=\frac{1}{h^4}(f_{i+2,j}-4f_{i+1,j}+6f_{i,j}-4f_{i-1,j}-f_{i-2,j}) \tag{21}$$

$$\frac{\partial^5 f}{\partial\eta^5}=\frac{1}{2h^5}(f_{i+3,j}-4f_{i+2,j}-3f_{i+1,j}-5f_{i-1,j}+4f_{i-2,j}-2f_{i-3,j}) \tag{22}$$

$$\frac{\partial^3 f}{\partial\eta^2\partial\tau}=\frac{1}{h^2k}(f_{i+1,j}-f_{i+1,j-1}-2f_{i,j}+2f_{i,j-1}+f_{i-1,j}-f_{i-1,j-1}) \tag{23}$$

$$\frac{\partial^4 f}{\partial\eta^3\partial\tau}=\frac{1}{2h^3k}\begin{pmatrix} f_{i+2,j}-f_{i+2,j-1}-2f_{i+1,j}+2f_{i+1,j-1} \\ +2f_{i-1,j}-2f_{i-1,j-1}-f_{i-2,j}+f_{i-2,j-1}\end{pmatrix} \tag{24}$$

$$\frac{\partial^5 f}{\partial\eta^2\partial\tau^3}=\frac{1}{h^2k^3}\begin{pmatrix} f_{i+1,j}-3f_{i+1,j-1}+f_{i+1,j-2}-f_{i+1,j-3}-2f_{i,j}+4f_{i,j-1} \\ -4f_{i,j-1}-4f_{i,j-2}+2f_{i,j-1}+f_{i-1,j-2}-f_{i-1,j-3}\end{pmatrix} \tag{25}$$

$$\frac{\partial^4 f}{\partial\eta^2\partial\tau^2}=\frac{1}{h^2k^2}\begin{pmatrix} f_{i+1,j}-2f_{i+1,j-1}-2f_{i,j}+f_{i+1,j-2} \\ +4f_{i,j-1}-2f_{i,j-2}+f_{i-1,j}-2f_{i-1,j-1}+f_{i-1,j-2}\end{pmatrix} \tag{26}$$

Equation (13) can be written as

$$\begin{aligned}
&\left(\frac{1}{k}\right)(f_{i,j}-f_{i,j-1})+\frac{\sqrt{2}d}{2h}(f_{i+1,j}-f_{i-1,j}) \\
&\quad=\frac{1}{2h^2}(f_{i+1,j}-2f_{i,j}+f_{i-1,j}) \\
&\qquad+\frac{a}{2}\left[\begin{array}{l}\frac{1}{h^2k}(f_{i+1,j}-f_{i+1,j-1}-2f_{i,j}+2f_{i,j-1}+f_{i-1,j}-f_{i-1,j-1}) \\ \quad+\frac{d}{2h^3}(f_{i+2,j}-2f_{i+1,j}+2f_{i-1,j}-f_{i-2,j})\end{array}\right] \\
&\qquad+\frac{b_1}{2}\left[\begin{array}{l}\frac{1}{h^2k}\begin{pmatrix} f_{i+1,j}-2f_{i+1,j-1}+f_{i+1,j-2}-2f_{i,j}+4f_{i,j-1} \\ -2f_{i,j-2}+f_{i-1,j}-2f_{i-1,j-1}+f_{i-1,j-2}\end{pmatrix} \\ \quad+\frac{d}{2kh^3}\begin{pmatrix} f_{i+2,j}-f_{i+2,j-1}-2f_{i+1,j}+2f_{i+1,j-1} \\ +2f_{i-1,j}-2f_{i-1,j-1}-f_{i-2,j}+f_{i-2,j-1}\end{pmatrix} \\ \quad+\frac{2d^2}{h^4}\begin{pmatrix} f_{i+2,j}-4f_{i+1,j}+6f_{i,j} \\ -4f_{i-1,j}+f_{i-2,j}\end{pmatrix}\end{array}\right] \\
&\qquad+\frac{3(b_2+b_3)}{2}\left[\frac{1}{2h^4}(f_{i+1,j}+f_{i-1,j})^2(f_{i+1,j}-2f_{i,j}+f_{i-1,j})\right]
\end{aligned}$$

$$+\frac{c_1}{2}\left[\begin{array}{l}\dfrac{1}{h^2k^3}\left(\begin{array}{l}f_{i+1,j}-3f_{i+1,j-1}+f_{i+1,j-2}-f_{i+1,j-3}\\ \quad -2f_{i,j}+4f_{i,j-1}-4f_{i,j-1}-4f_{i,j-2}+2f_{i,j-1}\\ \quad +f_{i-1,j-2}-f_{i-1,j-3}\end{array}\right)\\ +\dfrac{3d}{\sqrt{2}h^3k^2}\left(\begin{array}{l}f_{i+2,j}-2f_{i+2,j-1}+f_{i+2,j-2}-2f_{i+1,j}\\ \quad +4f_{i+1,j-1}-2f_{i+1,j-2}+2f_{i-1,j}-4f_{i-1,j-1}\\ \quad +2f_{i-1,j-2}-f_{i-2,j}+2f_{i-2,j-1}-f_{i-2,j-2}\end{array}\right)\\ +\dfrac{6d}{h^4k}\left(\begin{array}{l}f_{i+2,j}-f_{i+2,j-1}-4f_{i+1,j}+4f_{i+1,j-1}+6f_{i,j}\\ \quad -10f_{i,j-1}+4f_{i-1,j-1}+4f_{i-2,j}-f_{i-2,j-1}\end{array}\right)\\ +\dfrac{d^3}{\sqrt{2}h^5}\left(\begin{array}{l}f_{i+3,j}-4f_{i+2,j}-3f_{i+1,j}\\ \quad -5f_{i-1,j}+4f_{i-2,j}-2f_{i-3,j}\end{array}\right)\end{array}\right]$$

$$+\frac{(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)}{2}\left[\begin{array}{l}\dfrac{1}{4h^4k}(f_{i+1,j}-f_{i-1,j})\\ \quad (f_{i+1,j}-2f_{i,j}+f_{i-1,j})\\ \quad (f_{i+1,j}-f_{i+1,j-1}-f_{i-1,j}+f_{i-1,j-1})\\ +\dfrac{1}{4h^4k}(f_{i+1,j}-f_{i-1,j})^2\\ \left(\begin{array}{l}f_{i+1,j}-f_{i+1,j-1}-2f_{i,j}\\ \quad +2f_{i,j-1}+f_{i-1,j}-f_{i-1,j-1}\end{array}\right)\\ +\dfrac{2d}{h^3}(f_{i+1,j}-f_{i-1,j})\\ \quad (f_{i+1,j}-2f_{i,j}+f_{i-1,j})\\ +\dfrac{\sqrt{2}d}{8h^5}(f_{i+1,j}-f_{i-1,j})^2\\ \quad (f_{i+2,j}-2f_{i+1,j}+2f_{i-1,j}-f_{i-2,j})\end{array}\right] \qquad (27)$$

The above system of algebraic equations also gives

$$\begin{aligned}R_i=&\,Af_{i,j}+Bf_{i+1,j}+Cf_{i-1,j}+Df_{i+2,j}+Ef_{i-2,j}+E_1f_{i+3,j}\\ &+E_2f_{i-3,j}+K_1f_{i+1,j}^3+K_2f_{i+1,j}^2f_{i,j}+K_3f_{i+1,j}^2f_{i-1,j}+K_4f_{i-1,j}^2f_{i+1,j}\\ &+K_5f_{i-1,j}^2f_{i,j}+K_6f_{i-1,j}^3+K_7f_{i+1,j}f_{i-1,j}f_{i,j}+K_8f_{i+1,j}^2f_{i+1,j-1}\\ &+K_9f_{i+1,j}^2f_{i+1,j-1}+K_{10}f_{i+1,j}f_{i,j}f_{i+1,j-1}+K_{11}f_{i+1,j}f_{i-1,j-1}f_{i,j}\\ &+K_{12}f_{i-1,j}f_{i+1,j-1}f_{i,j}+K_{13}f_{i-1,j}f_{i,j}f_{i-1,j-1}+K_{14}f_{i+1,j}^2f_{i,j-1}\\ &+K_{15}f_{i-1,j}^2f_{i,j-1}+K_{16}f_{i+1,j}f_{i-1,j}f_{i+1,j-1}+K_{17}f_{i+1,j}f_{i-1,j}f_{i,j-1}\end{aligned}$$

$$+K_{18}f_{i+1,j}f_{i-1,j}f_{i-1,j-1}+K_{19}f_{i+1,j}^2+K_{20}f_{i+1,j}f_{i,j}$$
$$+K_{21}f_{i-1,j}^2+K_{22}f_{i-1,j}f_{i,j}+K_{23}f_{i+1,j}^2f_{i+2,j}+K_{24}f_{i+1,j}^2f_{i-2,j}$$
$$+K_{25}f_{i-1,j}^2f_{i+2,j}+K_{26}f_{i-1,j}^2f_{i-2,j}+K_{27}f_{i+1,j}f_{i-1,j}f_{i+2,j}$$
$$+K_{28}f_{i+1,j}f_{i-1,j}f_{i-2,j}+Ff_{i,j-1}+Gf_{i+1,j-1}+Hf_{i-1,j-1}$$
$$+If_{i+2,j-2}+Jf_{i-2,j-1}+Kf_{i+1,j-2}+Lf_{i,j-2}+Mf_{i-1,j-2}$$
$$+Nf_{i-1,j-3}+Pf_{i+1,j-3}+Qf_{i+2,j-2}+Rf_{i-2,j-2} \tag{28}$$

where

$$A=\left[\frac{1}{k}+\frac{1}{h^2}+\frac{a}{h^2k}+\frac{b_1}{h^2k^2}-\frac{6b_1d^2}{h^4}+\frac{c_1}{h^2k^3}\right]$$

$$B=\left[\begin{array}{l}-\dfrac{d}{\sqrt{2}h}-\dfrac{1}{2h^2}-\dfrac{a}{2h^2k}+\dfrac{ad}{2h^3}-\dfrac{b_1}{2h^2k^2}+\dfrac{b_1d}{2kh^3}\\ \quad+\dfrac{4b_1d^2}{h^4}-\dfrac{c_1}{2h^2k^3}+\dfrac{3dc_1}{\sqrt{2}k^2h^3}+\dfrac{3c_1d^3}{2\sqrt{2}h^5}\end{array}\right]$$

$$C=\left[\begin{array}{l}-\dfrac{d}{\sqrt{2}h}-\dfrac{1}{2h^2}-\dfrac{a}{2h^2k}-\dfrac{ad}{2h^3}-\dfrac{b_1}{2h^2k^2}-\dfrac{b_1d}{2kh^3}\\ \quad+\dfrac{4b_1d^2}{h^4}-\dfrac{3dc_1}{\sqrt{2}k^2h^3}+\dfrac{5c_1d^3}{2\sqrt{2}h^5}\end{array}\right]$$

$$D=\left[-\frac{ad}{4h^3}-\frac{b_1d}{4h^3k}-\frac{b_1d^2}{h^4}-\frac{3dc_1}{2\sqrt{2}k^2h^3}+\frac{\sqrt{2}c_1d^3}{h^5}\right]$$

$$E=\left[\frac{ad}{4h^3}+\frac{b_1d}{4h^3k}-\frac{b_1d^2}{h^4}+\frac{3dc_1}{2\sqrt{2}k^2h^3}-\frac{\sqrt{2}c_1d^3}{h^5}\right]$$

$$E_1=-\frac{c_1d^3}{2\sqrt{2}h^5},\quad E_2=\frac{c_1d^3}{\sqrt{2}h^5}$$

$$K_1=-\frac{3(b_2+b_3)}{8h^4}-\frac{1}{2h^4k}(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)$$
$$-\frac{d}{4h^5}(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)$$

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$$
\begin{aligned}
K_2 &= \frac{3(b_2+b_3)}{4h^4}+\frac{1}{2h^4k}(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)\\
K_3 &= \frac{3(b_2+b_3)}{8h^4}+\frac{1}{2h^4k}(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)\\
&\quad -\frac{3d}{4h^5}(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)\\
K_4 &= \frac{3(b_2+b_3)}{8h^4}+\frac{1}{4h^4k}(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)\\
&\quad +\frac{3d}{4h^5}(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)\\
K_5 &= \frac{3(b_2+b_3)}{2h^4}\\
K_6 &= -\frac{3(b_2+b_3)}{8h^4}+\frac{1}{4h^4k}(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)\\
&\quad -\frac{d}{4h^5}(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)\\
K_7 &= -\frac{3(b_2+b_3)}{2h^4}-\frac{1}{2h^4k}(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)\\
K_8 &= \frac{1}{4h^4k}(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)\\
K_9 &= K_8, \quad K_{10}=-K_8, \quad K_{11}=K_8\\
K_{12} &= -K_8, \quad K_{13}=K_8, \quad K_{14}=-K_8\\
K_{15} &= -K_8, \quad K_{16}=K_{15}, \quad K_{17}=-2K_{15}\\
K_{18} &= K_{15}, \quad K_{19}=\frac{d}{h^3}(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)\\
K_{20} &= 4K_{19}, \quad K_{21}=K_{19}, \quad K_{22}=-K_{20}\\
K_{23} &= -\frac{d}{8h^5}(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)\\
K_{24} &= -\frac{d}{4h^5}(6c_2+2c_3+2c_4+2c_5+6c_7+2c_8)
\end{aligned} \tag{29}
$$

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$$K_{25}=K_{23},\quad K_{26}=-K_{23},\quad K_{27}=-K_{24}$$

$$K_{28}=-K_{24},\quad F=-\frac{1}{k}-\frac{a}{h^2k}-\frac{2b_1}{h^2k^2}-\frac{3c_1}{h^2k^3}$$

$$G=\frac{a}{2h^2k}+\frac{b_1}{h^2k^2}-\frac{b_1d}{2kh^3}+\frac{3c_1}{2h^2k^3}-\frac{3\sqrt{2}dc_1}{k^2h^3}$$

$$H=\frac{a}{2h^2k}+\frac{b_1}{h^2k^2}+\frac{b_1d}{2kh^3}+\frac{3\sqrt{2}dc_1}{k^2h^3}$$

$$I=\frac{b_1d}{4h^3k}+\frac{3dc_1}{\sqrt{2}h^3k^2},\quad J=-\frac{b_1d}{4h^3k}-\frac{3dc_1}{\sqrt{2}h^3k^2}$$

$$K=-\frac{b_1}{2h^2k^2}-\frac{c_1}{2h^2k^3}+\frac{3dc_1}{\sqrt{2}h^3k^2}$$

$$L=\frac{b_1}{h^2k^2}+\frac{2c_1}{h^2k^3}$$

$$M=-\frac{b_1}{2h^2k^2}-\frac{c_1}{2h^2k^3}-\frac{3dc_1}{\sqrt{2}h^3k^2}$$

$$N=\frac{c_1}{2h^2k^3},\quad P=N$$

$$Q=-\frac{3dc_1}{2\sqrt{2}h^3k^2},\quad R=-Q$$

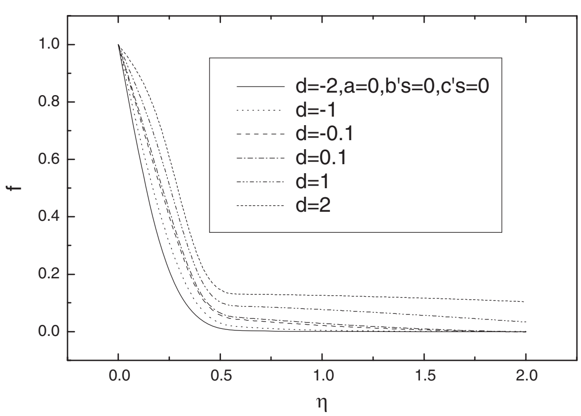

Figure 1. Influence of suction/blowing on the velocity distribution for the Newtonian fluid.

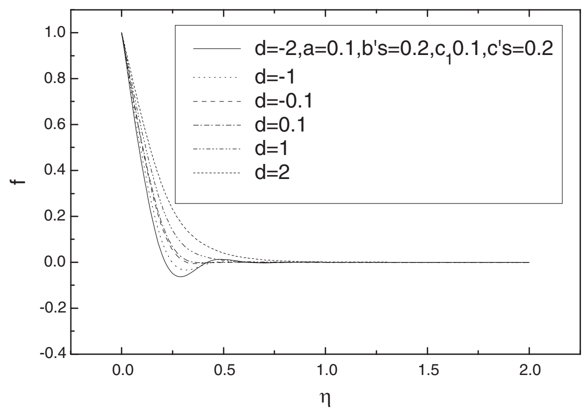

Figure 2. Influence of suction/blowing on the velocity distribution for the fourth-order fluid.

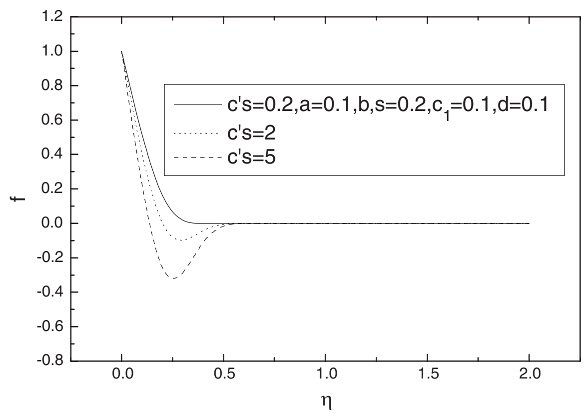

Figure 3. Variation in the fourth-order parameters c_i $(i=2\text{–}8)$ on f.

Now the initial and boundary conditions can be written in the following form:

$$f_{0,j}=1, \quad f_{M,j}=0, \quad f_{i,0}=0, \quad i=0,1,2,\ldots,M, \quad j=0,1,2,3\ldots \tag{30}$$

Here M denotes an integer large enough such that Mh approximates infinity. The augmented boundary conditions in terms of f are

$$\begin{aligned}\frac{\partial f(\infty,\tau)}{\partial\eta}&=0\\ \frac{\partial^2 f(\infty,\tau)}{\partial\eta^2}&=0\\ \frac{\partial^3 f(\infty,\tau)}{\partial\eta^3}&=0\end{aligned} \tag{31}$$

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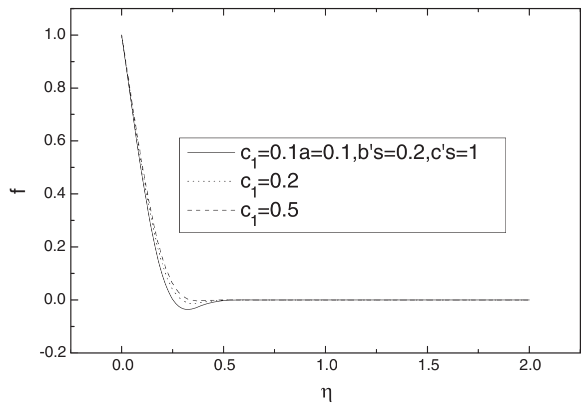

Figure 4. Variation in the fourth-order parameter c_1 on f.

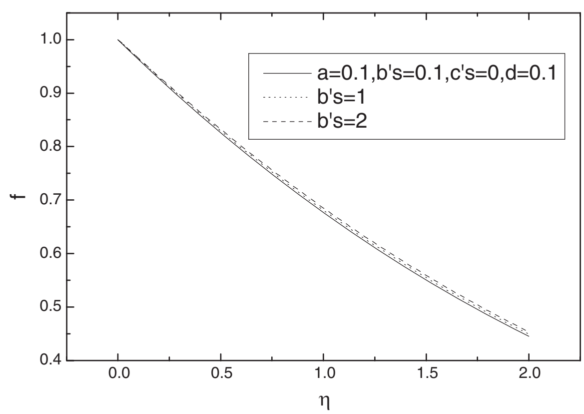

Figure 5. Variation in the third-grade parameters on f.

and consequently the problem becomes well posed. These boundary conditions are discretized to give

$$\frac{f_{M+1,j}-f_{M,j}}{h}=0$$

i.e.

$$f_{M+1,j}=f_{M,j} \tag{32}$$

The system consisting of Equations (28)–(32) has been solved numerically by employing Newton's method [22]. Solutions for the non-Newtonian fluid models are obtained for $\tau=2\pi$. From the numerical solution f is used to express the non-dimensional velocity profile parallel to x-axis. Results for the flow are obtained for various values of the involving parameters.

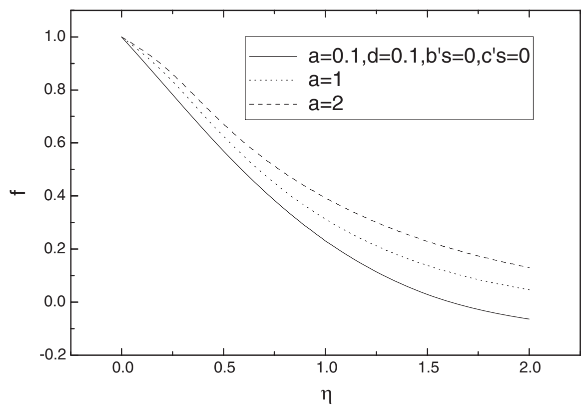

Figure 6. Variation in the second-grade parameters on f.

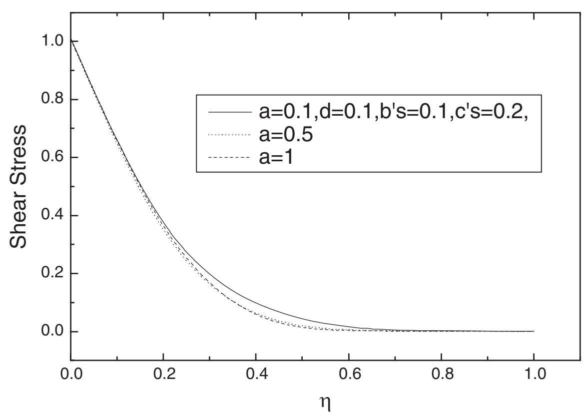

Figure 7. Variation in the second-grade parameters on shear stress.

The influence of suction and blowing on the velocity f is shown in Figure 1. This Figure shows the variation in d for the case of the Newtonian fluid. Here it is noted that suction causes reduction in the boundary layer thickness whereas blowing increases the layer thickness.

In order to illustrate the influence of suction/blowing on f in the case of fourth-grade fluid, we made Figure 2. This Figure elucidates the similar characteristics as in Figure 1. However, it is found that boundary layer thickness in case of fourth-order fluid is larger than that of Newtonian fluid. Figure 3 has been plotted just to see the variation of γ_i $(i=2\text{–}8)$ on f when other parameters in the fourth-order fluid are fixed. It is revealed that boundary layer thickness decreases by increasing γ_i $(i=2\text{–}8)$. Figure 4 shows the variation in the fourth-order parameter c_1 on f. Here it is observed that f increases by increasing c_1. Figures 5 and 6 indicate the variation in f in third- and second-order fluids, respectively. These figures show that boundary layer thickness in third-order fluid is less than that in the second-order fluid. However, the boundary layer in both the fluids is less when compared with fourth-order fluid. Figures 7–9 represent the non-dimensional shear stress for the

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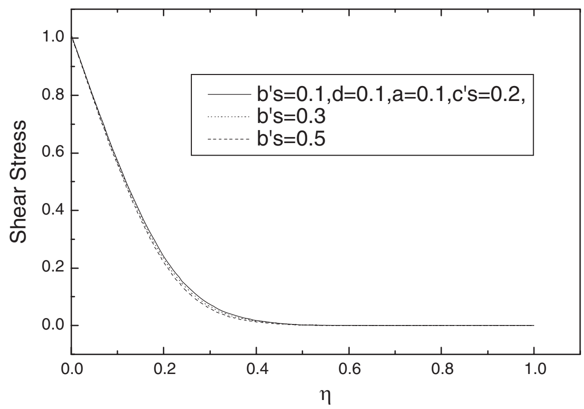

Figure 8. Variation in the third-grade parameters on shear stress.

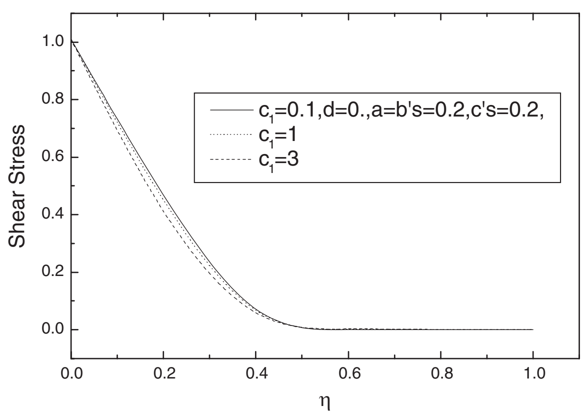

Figure 9. Variation in the fourth-order parameter c_1 on shear stress.

different values of second-, third- and fourth-order parameters. These figures indicate that increase in these parameters results in decrease in the shear stress (Figure 10).

4. CONCLUDING REMARKS

The effects of suction and blowing on the flows of an incompressible Newtonian and non-Newtonian fluid have been studied. The governing equation with the boundary and initial conditions has been non-dimensionalized. Numerical solution of the nonlinear problem has also been obtained using Newton's method. From the present investigation, it may be concluded that the boundary layer thickness decreases owing to an increase in the suction parameter where as in blowing case it increases when compared with suction. The boundary layer thickness in fourth-order fluid is larger than that in Newtonian fluid. The results for Newtonian, second-grade and third-grade fluid models

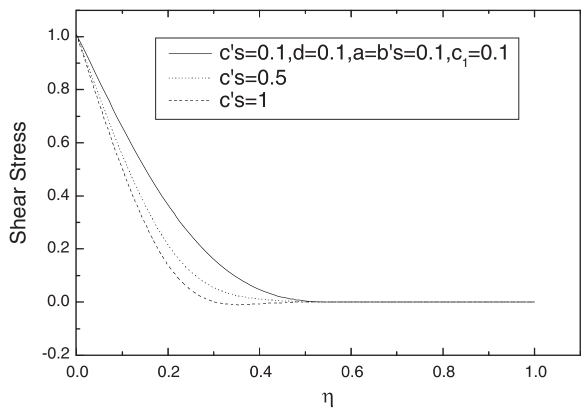

Figure 10. Variation in the fourth-order parameters c_i $(i = 2\text{–}8)$ on shear stress.

can be recovered as the limiting cases of the present solution by taking appropriate values of the material constants.

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